A Class of Univalent Analytic Functions Involving the Ruscheweyh Derivatives Operator and Hadamard Product

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This section is devoted to univalent mappings defined hadamard product involving a Ruscheweyh Derivatives Op_{tor} .

Let $\mathcal{A}$ stand for the class of mapping

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

whichever regular and one to one in the $\mathcal{U}_D$ $\mathcal{U} = \{z/|z| < 1\}$

For $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ (1.2)

the Hd_{pro} of f(z) & g(z) be

$$(f * g) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \quad z \in \mathcal{U} \tag{1.3}$$

Let D^mf(z) indicate the m[th] order derivative

The Ruschweyh derivative is definite go with D^m:S→ S like wise

$$D^m f(z) = \frac{z}{(1-z)^{m+1}} * f(z), \qquad m > -1$$

$$= \frac{z\left(z^{m-1} f(z)\right)^m}{m!} \qquad m \in N_0 = 0,1,2$$

$$= z + \sum_{m=2}^{\infty} a_k B(m,k) z^k \tag{1.4}$$

wherever B(m,k) = $\binom{m+k-1}{m}$

note that $\frac{z}{(1-z)^{m+1}} = z + \sum_{m=2}^{\infty} a_k B(m,k) z^k$ where m> -1

as well as $D^0 f(z) = f(z), D' f(z) = z f'(z)$

Now f∈ $\mathcal{A}$ & satisfy

$$\left| \frac{z\frac{D^{m+1} f(z)}{D^m f(z)} - 1}{(1-\alpha)\mu + \alpha\lambda - \lambda z \frac{D^{m+1} f(z)}{D^m f(z)}} \right| < \delta \tag{1.5}$$

For $0 \le \alpha < 1, \quad 0 < \delta \le 1, \quad 0 \le \lambda < \mu \le 1$

The study here is inspired by S. Khairnar & M.More [14]

1.1 Coefficient Inequality

Theorem **1:**Let f∈ S , Then f∈ $H(\alpha, \delta, \mu, \lambda)$ iff.

$$\sum_{k=2}^{\infty} \left[(k-1) + \delta\left((k-\alpha)\lambda - (1-\alpha)\mu\right)\right] B(m,k) a_k \le \delta(1-\alpha)(\mu - \lambda) \tag{1.6}$$

Proof : Suppose f(z)∈ $H(\alpha,\delta,\mu,\lambda)$ by (1.4)

$$\left|\frac{z\frac{D^{m+1}f(z)}{D^m f(z)}-1}{(1-\alpha)\mu+\alpha\lambda-\lambda z\frac{D^{m+1}f(z)}{D^m f(z)}}\right|<\delta$$

$$D^m f(z)=z+\sum_{m=2}^{\infty}a_k B(m,k)z^k$$

$$D^{m+1} f(z)=1+\sum_{m=2}^{\infty}ka_k B(m,k)z^{k-1}$$

$$zD^{(m+1)}f(z)-D^m f(z)=z+\sum_{m=2}^{\infty}ka_k B(m,k)z^k-z-\sum_{m=2}^{\infty}a_k B(m,k)z^k$$

$$=\sum_{k=2}^{\infty}(k-1)a_k B(m,k)z^k$$

Now

$$(1-\alpha)\mu+\alpha\lambda-\lambda z\frac{D^{m+1}(z)}{D^m(z)}=[(1-\alpha)\mu+\alpha\lambda]D^m(z)-\lambda zD^{m+1}(z)$$

$$=[(1-\alpha)\mu+\alpha\lambda]\left[z+\sum_{m=2}^{\infty}a_k B(m,k)z^k\right]-\lambda z\left[1+\sum_{m=2}^{\infty}ka_k B(m,k)z^{k-1}\right]$$

$$=[(1-\alpha)\mu+\alpha\lambda-\lambda]z+\sum_{m=2}^{\infty}[(1-\alpha)\mu+\alpha\lambda-\lambda k]\,a_k B(m,k)z^k$$

$$=\left[(1-\alpha)\mu-(1-\alpha)\lambda\right]z+\sum_{m=2}^{\infty}\left[(1-\alpha)\mu-(k-\alpha)\lambda\right]a_k B(m,k)z^k$$

$$=\left[(1-\alpha)(\mu-\lambda)\right]z+\sum_{m=2}^{\infty}\left[(1-\alpha)\mu-(k-\alpha)\lambda\right]a_k B(m,k)z^k$$

$$\left|\frac{\sum_{m=2}^{\infty}(k-1)a_k B(m,k)z^k}{\left[(1-\alpha)(\mu-\lambda)\right]z+\sum_{m=2}^{\infty}\left[(1-\alpha)\mu-(k-\alpha)\lambda\right]a_k B(m,k)z^k}\right|<\delta \qquad (1.7)$$

We be familiar with $|Re(z)|<|z|$

$$\text{Re}\left|\frac{\sum_{m=2}^{\infty}(k-1)a_k B(m,k)z^k}{\left[(1-\alpha)(\mu-\lambda)\right]z+\sum_{m=2}^{\infty}\left[(1-\alpha)\mu-(k-\alpha)\lambda\right]a_k B(m,k)z^k}\right|<\delta$$

We choose values of greater than expression & allowing z→ 1 throughout the $\mathcal{R}e_{al}$ value we acquire

$$\sum_{m=2}^{\infty}(k-1)a_k B(m,k) \le \delta\left[(1-\alpha)(\mu-\lambda)+\sum_{m=2}^{\infty}\left[(1-\alpha)\mu-(k-\alpha)\lambda\right]a_k B(m,k)\right]$$

$$\therefore \sum_{m=2}^{\infty}[(k-1)+\delta[(k-\alpha)\lambda-(1-\alpha)\mu]a_k\, B(m,k) \le \delta(1-\alpha)(\mu-\lambda)$$

Conversely

Suppose that

$$\sum_{m=2}^{\infty}[(k-1)+\delta[(k-\alpha)\lambda-(1-\alpha)\mu]a_k\, B(m,k) \le \delta(1-\alpha)(\mu-\lambda)$$

Without help enclose

$|zD^{m+1}f(z)-D^m f(z)|-\delta|(1-\alpha)\mu+\alpha\lambda-\lambda z D^{m+1}f(z)|<0$

With the position

$$\left|\sum_{m=2}^{\infty}(k-1)a_k B(m,k)z^k\right|-\delta\left|(1-\alpha)(\mu-\lambda)z+\sum_{m=2}^{\infty}(1-\alpha)\mu(k-\lambda)a_k B(m,k)z^k\right|$$

For $|z|<r<1$ the condition is bounded above

$$\sum_{m=2}^{\infty}(k-1)a_k\, B(m,k)r^k$$

$$-\delta\left[(1-\alpha)(\mu-\lambda)z+\sum_{m=2}^{\infty}(1-\alpha)\mu(k-\lambda)a_k B(m,k)z^k\right]<0$$

$$\sum_{m=2}^{\infty}\left[(k-1)+\delta\left[(k-\alpha)\lambda-(1-\alpha)\mu\right]a_k B(m,k)\le\delta(1-\alpha)(\mu-\lambda)\right.$$

$\therefore$ f(z)∈ $H(\alpha,\delta,\mu,\lambda)$

Corollary **1:** If f(z)∈ $H(\alpha,\delta,\mu,\lambda)$ then

$$a_k \le \frac{\delta(1-\alpha)(\mu-\lambda)}{\left[(k-1)+\delta\left((k-\alpha)\lambda-(1-\alpha)\mu\right)\right]B(m,k)} \tag{1.8}$$

and equality holds for

$$f(z)=z+\frac{\delta(1-\alpha)(\mu-\lambda)}{\left[(k-1)+\delta\left((k-\alpha)\lambda-(1-\alpha)\mu\right)\right]B(m,k)}z^k$$

4.2.2 Growth And Distortion Theorem

Theorem **2 :**Whenever the mapping f(z)∈ $H(\alpha,\delta,\mu,\lambda)$ then

$$|z|-\frac{\delta(1-\alpha)(\mu-\lambda)}{\left[1+\delta\left((2-\alpha)\lambda-(1-\alpha)\mu\right)\right]B(m,2)}|z|^2 \le |f(z)|$$
$$\le |z|+\frac{\delta(1-\alpha)(\mu-\lambda)}{\left[1+\delta\left((2-\alpha)\lambda-(1-\alpha)\mu\right)\right]B(m,2)}|z|^2 \tag{1.9}$$

With the equality for

$$f(z)=z+\frac{\delta(1-\alpha)(\mu-\lambda)}{[1+\delta((2-\alpha)\lambda-(1-\alpha)\mu)]B(m,2)}z^2 \qquad (1.10)$$

Proof **:** from theorem (1) f(z)∈ $H(\alpha,\delta,\mu,\lambda)$ iff

$$\sum_{m=2}^{\infty}[(k-1)+\delta((k-\alpha)\lambda-(1-\alpha)\mu)]B(m,k)a_k \le \delta(1-\alpha)(\mu-\lambda)$$

Now

$$|f(z)| \le |z| + a_k|z|^k$$

$$|f(z)| \le |z| + |z|^2\sum_{k=2}^{\infty}|a_k|$$

$$|f(z)| \le |z| + |z|^2\frac{\delta(1-\alpha)(\mu-\lambda)}{[1+\delta((2-\alpha)\lambda-(1-\alpha)\mu)]B(m,2)}$$

$$|f(z)| \le |z| + \frac{\delta(1-\alpha)(\mu-\lambda)}{[1+\delta((2-\alpha)\lambda-(1-\alpha)\mu)]B(m,2)}|z|^2 \qquad (1.11)$$

Simillary

$$|f(z)| \ge |z| - \sum_{k=2}^{\infty}a_k|z|^k$$

$$|f(z)| \ge |z| - |z|^2\sum_{k=2}^{\infty}|a_k|$$

$$|f(z)| \ge |z| - |z|^2\frac{\delta(1-\alpha)(\mu-\lambda)}{[1+\delta((2-\alpha)\lambda-(1-\alpha)\mu)]B(m,2)}$$

$$|f(z)| \ge |z| - \frac{\delta(1-\alpha)(\mu-\lambda)}{[1+\delta((2-\alpha)\lambda-(1-\alpha)\mu)]B(m,2)}|z|^2 \qquad (1.12)$$

By (1.11) and (1.12) we obtain

$$|z| - \frac{\delta(1-\alpha)(\mu-\lambda)}{[1+\delta((2-\alpha)\lambda-(1-\alpha)\mu)]B(m,2)}|z|^2 \le |f(z)|$$

$$\le |z| + \frac{\delta(1-\alpha)(\mu-\lambda)}{[1+\delta((2-\alpha)\lambda-(1-\alpha)\mu)]B(m,2)}|z|^2$$

with the equality for

$$f(z)=z+\frac{\delta(1-\alpha)(\mu-\lambda)}{[1+\delta((2-\alpha)\lambda-(1-\alpha)\mu)]B(m,2)}z^2$$

which complete the proof.

Theorem **3:** If the mapping f(z)∈ $H(\alpha,\delta,\mu,\lambda)$ then

$$1-\frac{2\delta(1-\alpha)(\mu-\lambda)}{\left[1+\delta\left((2-\alpha)\lambda-(1-\alpha)\mu\right)\right]B(m,2)}|z| \le |f'(z)| \le 1+\frac{2\delta(1-\alpha)(\mu-\lambda)}{\left[1+\delta\left((2-\alpha)\lambda-(1-\alpha)\mu\right)\right]B(m,2)}|z| \quad (1.13)$$

The equality hold for

$$f(z) = z + \frac{2\delta(1-\alpha)(\mu-\lambda)}{\left[1+\delta\left((2-\alpha)\lambda-(1-\alpha)\mu\right)\right]B(m,2)}z^2 \quad (1.14)$$

Proof **:** From theorem (1) f(z)∈ $H(\alpha,\delta,\mu,\lambda)$ if and only if

$$\sum_{m=2}^{\infty}\left[(k-1)+\delta\left((k-\alpha)\lambda-(1-\alpha)\mu\right)\right]B(m,k)a_k \le \delta(1-\alpha)(\mu-\lambda)$$

Now

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k$$

$$f'(z)=1+\sum_{k=2}^{\infty}ka_k z^{k-1}$$

$$|f'(z)| \le 1+\frac{2\delta(1-\alpha)(\mu-\lambda)}{\left[1+\delta\left((2-\alpha)\lambda-(1-\alpha)\mu\right)\right]B(m,2)}|z|$$

Simillarly

$$|f'(z)| \ge 1-\frac{2\delta(1-\alpha)(\mu-\lambda)}{\left[1+\delta\left((2-\alpha)\lambda-(1-\alpha)\mu\right)\right]B(m,2)}|z|$$

$$\therefore 1-\frac{2\delta(1-\alpha)(\mu-\lambda)}{\left[1+\delta\left((2-\alpha)\lambda-(1-\alpha)\mu\right)\right]B(m,2)}|z| \le |f'(z)| \le 1+\frac{2\delta(1-\alpha)(\mu-\lambda)}{\left[1+\delta\left((2-\alpha)\lambda-(1-\alpha)\mu\right)\right]B(m,2)}|z|$$

The equality hold for

$$f(z) = z + \frac{2\delta(1-\alpha)(\mu-\lambda)}{\left[1+\delta\left((2-\alpha)\lambda-(1-\alpha)\mu\right)\right]B(m,2)}z^2$$

which complete the proof.

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