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Analytical Study of Generalized Series Equations Involving Special Functions Using Fractional Calculus Approaches

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Abstract

This investigates the properties of series equations involving special functions with the help of fractional calculus. Extending classical differential methods to non-integer order calculus allows the research to find more accurate and flexible ways to solve complex equations from engineering, physics and mathematics. Using special functions like Mittag-Leffler, Bessel and hyper geometric functions, scientists can present solutions in series form which reflects better the memory and hereditary qualities of many problems. It also discusses important methods such as the Laplace Transform Method, A domain Decomposition Method and Homotopy Analysis Method that are helpful in finding the solutions to fractional-order differential equations. The use of special functions with these techniques widens both the scope and use of analytical models, giving a strong basis for further research in advanced modelling.

Keywords: Fractional Calculus, Generalized Series Equations, Special Functions, Mittag-Leffler Function, Analytical Methods, Laplace Transform, Adomian.

1. INTRODUCTION

Research on generalized series equations has increased remarkably in recent times because of their widespread uses in mathematics, physics and engineering. Many times, these branches of mathematics include functions such as the Mittag-Leffler function, Bessel functions and hyper geometric functions which are important for solving challenging differential and integral equations in real systems. Exploring equations using traditional calculus is very effective, but in nature and physics, many things happen through memory and distance effects which are better explained using fractional calculus.

Instead of sticking to integer (whole number) orders, fractional calculus makes it possible to use fractions in differentiation and integration. The general framework allows for more effective analysis of systems with unusual brownian motion, viscoelastic components and signal processing. Fractional calculus greatly contributes to learning more about generalized series and how they behave and converge, when combined with special functions.

Mixing generalized series with fractional differential operators lets us handle and manipulate complex functions, mainly in the context of FDEs modelling and solving. These FDEs are often solved by using special functions like the Mittag-Leffler function which act as a natural change from exponential functions in the context of fractions.

The main purpose of this study is to use fractional calculus tools like the Riemann–Liouville, Caputo and Hadamard types of derivatives to analyze series that feature special functions. One purposes to find useful equations, define the rules for a series to converge and observe the features of the solutions in this setting. In addition, emphasis is placed on how fractional-order models can be used in fields like mathematical physics, control theory and signal processing which now rely more on them.

2. LITERATURE REVIEW

Kiryakova (2021) explained special functions and how they appear in fractional calculus. It found that Mittag-Leffler functions, Fox H-functions and Wright functions are necessary for solving fractional differential equations. Showing that they could be used like classical functions, these functions allow for better representation of anomalous diffusion and viscoelastic phenomena.

Singh, Srivastava, and Pandey (2023) edited a book containing studies on several special functions and the part they play in fractional calculus and engineering fields. This text discussed techniques from many different fields and explained how special functions find use in control theory, fluid dynamics and signal processing. It explained how fractional models are playing a larger role in solving difficult engineering problems.

Luchko (2021) expanded the field by showing how to calculate special cases using convolution series in fractional calculus. His studies greatly increased the understanding of fractional calculus by introducing several types of new integral transforms and convolution formulas. Understanding these functions, it was also shown they can be applied to boundary value problems and fractional partial differential equations.

Luchko (2021) developed a way to do the general fractional derivative algebraically. It merged various fractional derivatives and introduced methods to work with a variety of fractional operators. The research pointed out that using various methods is helpful for dealing with linear and nonlinear fractional differential equations in physics and engineering.

Anastassiou (2021) contributed to theories by introducing generalized fractional calculus. His studies included abstract fractional operators which he explained in terms of their properties, inequalities and how to approximate them. Based on the research, it became possible to study other aspects of numerics in fractional systems.

Baleanu et al. (2023) looked into the connection between fractional calculus, differential equations and neural networks. This study featured fractional-order systems and pointed out that using fractional differential equations along with machine learning could boost prediction abilities. In addition, researchers used numbers and simulations to test how these hybrid models can be used in different science fields.

Luchko (2023) Studied in detail general fractional integrals and derivatives and examined their numerous uses. It presented new fractional operators that improved classical fractional calculus using generalized kernels and different types of integrals. Luchko examined these operators and proved that using them can help solve different nonlinear phenomena described by fractional differential equations. It pointed out that generalized fractional operators can describe complicated behaviours that classical models with whole-number orders cannot handle well.

Oumarou et al. (2021) Using functions as input, worked with fractional calculus with analytic kernels, as another general way to describe fractional operators. By introducing analytic kernel functions, they came up with new ways to define fractional integrals and derivatives which led to improved ways to represent memory and hereditary aspects in various systems. They examined mathematical ideas, their characteristics and the ways they could be used in computation and applied areas. They included demonstrations outlining cases where these generalized operators solved anomalous diffusion and various fractional problems that occur.

3. FUNDAMENTALS OF FRACTIONAL CALCULUS AND SPECIAL FUNCTIONS.

It provides the key math foundations important for exploring the analysis of generalized series with special functions using fractional calculus. Basic concepts, definitions and main properties of fractional calculus are included, as well as common special functions in this field.

3.1. Introduction to Fractional Calculus

Fractional calculus is an extension of traditional calculus, where differentiation and integration are done for non-whole (fractional) numbers. Unlike pure derivatives and integrals of integer order that are usually used, fractional derivatives help describe the memory and hereditary effects of many systems that standard derivatives/integrals cannot capture.

- **Historical Background**

The study of fractional calculus or calculus for non-integer orders, is centuries old and very interesting. The Marquis de l'Hôpital asked Leibniz, who co-wrote classical calculus, a question in 1695 and Leibniz answered by introducing the notion of speed and rate. The Marquis questioned what could happen if the order of a derivative were just 1/2. Leibniz, being interested in the theory, felt it might produce a paradox, but he also thought about its benefits. It was at this point that what we now call fractional derivatives first appeared in writings.

Even though mathematicians became interested in fractions, the field of fractional calculus progressed very slowly due to missing tools and motivations. Joseph Liouville, Bernhard Riemann and Augustin-Louis Cauchy were among the mathematicians in the 19th century who initiated the mathematization of analysis. Liouville came up with a solid framework for fractional integration and put forward a derivative now called the Liouville fractional derivative. He improved these concepts by proposing the Riemann–Liouville integral and derivative which turned out to be crucial for the field.

The study of fractional calculus became more popular in the 20th century, mainly because of its possible uses in engineering, physics and biology. Miller, Ross and Caputo, along with other mathematicians, made contributions to the theory and introduced the Caputo derivative which became very widely used in applied sciences because its initial conditions fit well with physical questions. Lately, the field has developed a lot and been used in several areas, including anomalous diffusion, viscoelasticity, electrical networks, control theory, biomedical engineering and finance, making fractional calculus an important branch for both research and applications.

- **Definition of Fractional Derivatives and Integrals**

Ordinary calculus only involves integer-order differentiation and integration, while fractional calculus considers non-integer (fractional) orders which makes it suitable for more precise descriptions of certain dynamic systems. A variety of definitions of fractional derivatives and integrals are available, each suited for certain kinds of mathematical problems. The most used definitions say that:

- **Riemann–Liouville Fractional Integral and Derivative**

The fractional integral of order $\alpha>0$ for a function f(t) is defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau,$$

where $\Gamma(\cdot)$is the Gamma function. The corresponding derivative is obtained by differentiating the fractional integral

Caputo Fractional Derivative: Often preferred in initial value problems, it modifies the Riemann–Liouville derivative to allow for standard initial conditions:

$$D^{\alpha} f(t) = I^{n-\alpha} \frac{d^n}{dt^n} f(t),$$

where $n-1<\alpha<n$.

- **Grünwald–Letnikov and Other Definitions:** These provide equivalent but numerically friendly formulations of fractional derivatives.

- **Key Properties**

Fractional derivatives do not just look at the value of a function at a specific point; they also use the entire past of the function. Because of this trait, they work well in describing things that store information.

2. Special Functions in Mathematical Analysis

Many problems in applied mathematics, physics and engineering involve solving special functions which appear as the results of differential equations and integral equations. Elements of calculus are generalized in them and famous functions like Bessel functions, hypergeometric functions and Mittag-Leffler functions are included as well.

- **Bessel Functions:** Commonly found in wave and static potential problems, solutions to Bessel's differential equation. Known as $J\nu(x)$ with ν being its order.
- **Mittag-Leffler Function:** An extension of the exponential function which is very important in fractional calculus, often relating to fractional differential equations. Defined as.

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)},$$

where α>0. It describes fractional relaxation and anomalous diffusion phenomena.

- **Hypergeometric Functions:** There are many functions in this family, created by a series expansion and they all follow hypergeometric differential equations which help solve numerous mathematical problems.
- **Other Related Functions:** The Wright function, Legendre functions and Airy functions are often seen in discussions of generalized series and fractional calculus.

3. Relevance of Fractional Calculus and Special Functions to Series Equations

Because of fractional derivatives and integrals, it is possible to extend the method of classical series solutions to fractional differential equations. Special functions frequently help make up the basic units inside the structure of generalized series expansions.

- **Series Representations:** Many such functions can be expanded into series which, by making use of fractional operators, can lead to solving problems.
- **Interplay with Fractional Operators:** It makes it possible to generalize traditional results by using special functions.

GENERALIZED SERIES EQUATIONS INVOLVING SPECIAL FUNCTIONS

Series equations are widely used in mathematics to show how complex functions or solutions to differential equations can be shown as infinite series. Besides traditional power series, these equations include special functions as the central elements of the series. With this strategy, it's possible to study events that are not always steady or constant and these kinds of events are usually found in quantum mechanics, fluid dynamics and signal processing.

Usually, in generalized series equations, the answer is shown as a mix of various special functions such as Bessel functions, Legendre polynomials, Hypergeometric functions or Mittag-Leffler functions, with each having its own coefficient that is found either through finding the answer or by looking at the boundary conditions. The form of a such a series could be described as:

$$y(x) = \sum_{n=0}^{\infty} a_n \phi_n(x),$$

where ϕn(x) stands for a unique function or family of functions developed from a identified collection of orthogonal functions. They are very useful because they fulfil important differential equations and commonly have important features like being orthogonal and repetitive which support both calculation and understanding of solutions.

Here, special functions matter because they are naturally suited to describe the solutions to many important and modern physical problems. Problems involving cylindrical symmetry often produce results with Bessel functions and Mittag-Leffler functions are crucial for solving equations with a memory effect or recall. When functions are added to series formulations, they become better adapted to match the characteristics of a particular physical problem than what polynomials or plain power series could offer.

Also, the combination of fractional calculus with generalized series increases their usefulness. Fractional derivatives make it possible to include each term of the special functions series which allows us to model anomalous diffusion, viscoelastic substances and similar situations that include memory effects. For this reason, the mathematical tools based on special functions and fractional derivatives form a vital way to solve real examples of complicated models.

4. ANALYTICAL SOLUTION TECHNIQUES USING FRACTIONAL CALCULUS

Fractional calculus methods allow for the effective solution of complex equations from the real world which often deal with memory, hereditary traits or unusual diffusion. Unlike classical methods which only look at integer-order derivatives and integrals, fractional calculus uses non-integer (fractional) orders, making analysis of different systems more detailed and flexible. Working with equations that have special functions is where the value of these techniques is most needed, since fractional operators may address issues not handled by standard models.

In fractional calculus, the Laplace transformation is one of the most widely used tools for analysis.

Transform Method This technique changes fractional differential equations from the time domain to the complex frequency domain which makes them simpler. Fractional derivatives in this area are written with powers of the Laplace variable s and this simplifies how the equations are handled. Solving the equation in the Laplace domain is followed by using the inverse Laplace transform to return the answer to the original domain. It works best when you need to solve linear fractional differential equations with Caputo or Riemann–Liouville derivatives and the Mittag-Leffler function.

The Adomian Decomposition Method (ADM) is also considered a powerful way to solve these problems. ADM is applied in solving both linear and nonlinear screamed fractional differential equations. Rather than using linearization, discretization or perturbation, it breaks the solution into functions that quickly approach it as a series. ADM makes it possible to easily include fractional derivatives and special functions in the decomposition series for fractional equations.

Analytical solutions to fractional differential equations can also be found using the Homotopy Analysis Method (HAM) and the Homotopy Perturbation Method (HPM). The methods create a smooth one-to-one mapping (homotopy) from the first guess to the final answer, helping to deal with various nonlinear cases. Homotopys include fractional derivatives and solutions are created as series that become precise when they converge. These methods are helpful when the classical methods cannot be used because the equation does not contain small parameters.

In addition, people often use the Mittag-Leffler function expansion method when the solution naturally appears as this function. This applies mainly to fractional-order linear systems, whose solutions are often given by Mittag-Leffler functions. They generalize the exponential function and are important for explaining the evolution of fractional systems. Investigating the Mittag-Leffler function by expanding it in series usually leads to problem solutions that can be easily explained physically.

Besides these, the Variational Iteration Method (VIM), the Fractional Power Series Method and Operational Matrix Techniques based on polynomials like Legendre or Chebyshev are also used to find solutions for fractional differential equations. They are best used for problems including boundary conditions or variable coefficients and they often depend on special functions to work out the solution quickly.

All in all, methods from fractional calculus have made it possible to solve a large variety of equations that, before, were unsolvable or very difficult to solve. They allow us to better understand what the solutions do and also supply equations for further analysis, modeling and simulation. Having special function properties through generalized series, they are necessary for use in mathematical modeling and applied sciences today.

5. CONCLUSION

Applying fractional calculus to the study of generalized series equations including special functions helps build a useful framework for solving a range of challenging differential equations used in science and engineering. Integrating functions such as Mittag-Leffler, Bessel and hypergeometric functions with series and using Laplace Transform, Adomian Decomposition and Homotopy Analysis Methods from fractional calculus makes it easier to determine memory effects, odd behavior at different points and singularities than was done by traditional techniques. Because of this fusion, it is easier to address problems with systems that show anomalous diffusion, viscoelasticity and other unusual behaviors. All in all, it allows for more solutions to mathematics problems and better understanding of the underlying structures shaping reality.

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