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SOME FIXED POINT THEOREMS FOR ISHIKAWA ITERATIONS

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Abstract

This paper focuses on the convergence of certain Ishikawa type iterations to fixed points of maps satisfying the contractive conditions defined in the earlier chapter. This paper embibes some fixed point theorems for contractive conditions using Ishikawa iterations established by Albert K. Kalinde and B.E. Rhoades, Kalishankar Tiwary and S.C. Debnath and Rhoades.

In 1992, Albert K. Kalinde and B.E. Rhoades successfully derived sufficient conditions for the coefficients of Ishikawa iteration process. They proved, if the Ishikawa iterates of a continuous self-map G (of the unit interval) converge, then they converge to a fixed print of G.

They derived these following results:

Theorem 1

Let G be a continuous self map of $L \equiv [0,1]$ so that the Ishikawa iterates $\{u_n\}$ converge,

1 If $\lim \inf \alpha_n > 0$ and $\lim \inf \beta_n = 0$, then $\{u_n\}$ converges to a fixed point of G.

2 If A is regular and $\lim \inf \beta_n = 1$, then $\{u_n\}$ converges to a fixed point of G^2.

Proof

(1) Let $\lim u_n = z$. Then, $\exists$ a subsequence $\{n_i\}$ of $\{n\}$ such that $\lim_i \beta_{ni} = 0$. Therefore $y_{ni} = (1-\beta_{ni}) + \beta_{ni} G u_{ni}$

$y_{ni} - u_{ni} = \beta_{ni} (Gu_{ni} - u_{ni})$.

Thus $\lim |y_{ni} - u_{ni}| \leq 2 \beta_{ni}$, which implies that $\lim_i y_{ni} = z$.

Because $u_{ni+1} - u_{ni} = \alpha_{ni} (Gy_{ni} - U_{ni})$ and

$\lim \inf \alpha_n \mid Gz - z \mid \leq 0$, Therefore, $Gz = z$.

(2) $\because$ $\lim \text{Sup}\, \beta_n \leq 1 = \lim \inf \beta_n$ and $\lim \beta_n = 1$.

$\therefore$ $y_n \to Gz$. By the continuity of G,

$Gy_n \to G^2z$. Since $u_n \to z$ and A is regular, therefore,

$G^2z = z$.

By the example given below we can prove that theorem (1) is not applicable in an arbitrary Banach space with conditions $\lim \inf \alpha_n < 0$ and $0 < \lim \inf \beta_n < 1$. Define u(t) as a continuous function on L (closed unit interval) with conditions given below and E is a space of u(t),

Conditions : $u(0)=0$, $u(1) = 1$, $0 \leq u(t) \leq 1$, $u_0 = u_0(t)=1$, $f(u)[t]=t\, u(t)$.

Using, $u_{n+1} = (1 - \alpha_n) u_n + \alpha_n Gy_n$, $y_n = (1-\beta_n) u_n + \beta_n Gu_n$, $n \geq 0$.

Choosing $\alpha_n = 2/3$, $\beta_n = ½$, we get

$$u_n = \frac{u_0\left(1+t+t^2\right)^n}{3^n},\quad y_n = \frac{u_0\left(1+t\right)\left(1+t+t^2\right)^n}{2.3^n}$$

$\therefore$ For each t, $\{u_n\}$ converges but $\{u_n\}$ has no fixed points.

Theorem 2

Let G be a continuous self map of L (closed unit interval) and $\{\alpha_n\}$, $\{\beta_n\}$ satisfy the conditions.

i) $\alpha_n \geq 0$, $\beta_n \leq 1$, $\forall$ n

ii) $\lim \beta_n = 0$

iii) $\Sigma \alpha_n = \infty$

Then $u_{n+1} = (1 - \alpha_n) u_n + \alpha_n G [(1-\beta_n) u_n + \beta_n Gu_n]$

converges to a fixed point of G.

Proof

First of all we shall prove that $\{u_n\}$ satisfying three conditions which follow its definition, converges.

Definition of u_n

$u_{n+1} = (1-\alpha_n)u_n+\alpha_n G[(1-\beta_n)u_n+\beta_n Gu_n]$, for $n \geq 0$ .... (1)

Conditions :

$\{\alpha_n\}$, $\{\beta_n\}$ satisfy the conditions

1 $\alpha_n \geq 0$, $\beta_n \leq 1$, $\forall$ n

2 $\lim \beta_n = 0$

3 $\Sigma\alpha_n \beta_n = \infty$ and $u_0 \in L$

Equation (1) can be modified in the form,

$u_{n+1} = (1-\alpha_n) u_n + \alpha_n Gy_n$, where $y_n = (1-\beta_n) u_n + \beta_n Gu_n$, $n \geq 0$ .... (2)

Let us consider the existence of integer k such that $Gx_k = x_k$. By equation (2), we get

$y_k = u_k$ which gives $u_{k+1} = u_k$. Therefore by induction,

$u_n = u_k$, $\forall$ $n \geq k$. Hence the sequence converges to u_k.

Now suppose that $Gx_n \neq x_n$, $\forall$ n. Because $\{u_n\}$ is contained in L. Therefore, the sequence $\{u_n\}$ has at least one limit point in L. Let, $\lim_n \inf u_n = \xi_1$ and $\lim_n \sup u_n = \xi_2$. Then $\xi_1 \leq \xi_2$. Taking $\xi_1 < \xi_2$, we shall prove that $\xi_1 \leq G\xi_1$ and $G\xi_2 \leq \xi_2$. These two inequalities are true if $\xi_1 = 0$ and $\xi_2 = 1$. When $\xi_2<1$, proof is achieved by contradiction. If $\xi_2 < G\xi_2$, by the continuity of G at ξ_2, there exists $\delta>0$ such that $u < Gu$, $\forall$ $u \in (\xi_2\ \delta, \xi_2 + \delta)$. Choosing $\delta<\xi_2-\xi_1$ and using condition (2), we have $\lim \sup_n u_n = \xi_2 = \lim \sup_n y_n$. By the definition of lim Sup, there exists a, $\delta > 0$ and n_0 such that

$u_n < \xi_2 + \delta$ and $y_n < \xi_2 + \delta$, $\forall$ $n \geq n_0$ .... (3)

$\because$ The subsequence $\{u_{n_k}\}$ of $\{u_n\}$ converging to ξ_2

$\therefore$ n_0 can be chosen in such way that $u_{n_k} \in (\xi_2-\delta, \xi_2+\delta)$, $\forall$ $k \geq k_0$ and $n_{k0} \geq n_0$. Taking, $A = \{n : n \geq n_0$ and $u_n \in (\xi_2 - \delta, \xi_2 + \delta)\}$ .... (4)

we get A is non empty.

Now we shall prove A is equivalent to a number N. Let us consider any element of A be n. Then $u_n < Gu_n$ giving $u_n \leq y_n$. By (3), we arrive at the conclusion that $y \in (\xi_2 - \delta, \xi_2 + \delta)$. Now we have $u_n \leq y_n < Gy_n$ and from (2), we get $u_{n+1} - u_n = \alpha_n (Gy_n - u_n) \geq 0$ which implies $u_n \leq u_{n+1}$. Because $n+1>n \geq n_0$, (3) gives $\xi_2-\delta < u_n \leq u_{n+1} < \xi_2 + \delta$ and $u_{n+1} \in(\xi_2 - \delta, \xi_2 + \delta)$. This shows that n+1 belongs to (4) and by induction A is equivalent to N. Hence $u_n \in (\xi_2 - \delta, \xi_2 + \delta)$; $\forall$ $n \geq n_0$. Because δ satisfies the condition $\delta<\xi_2-\xi_1$ or $\xi_1 < \xi_2 - \delta$, Then ξ_1 is not a limit point of $\{u_n\}$, which is a contradiction. Therefore, $G\xi_2 \leq \xi_2$. Similarly, for $\xi_1 > o$ and $G\xi_1 < \xi_1$, we get $\xi_1 \leq G\xi_1$.

Now we shall prove that each $u \in (\xi_1, \xi_2)$ is a fixed point of G. If possible, let $\overset{*}{u} \in (\xi_1, \xi_2)$ so that $\overset{*}{u} \in G\overset{*}{u}$. Because G is cont at $\overset{*}{u}$, therefore, $\exists$ $\delta > 0$ such that $u < Gu$, $\forall$ $u\in(\overset{*}{u}-\delta, \overset{*}{u}+\delta)$ where δ is taken so that $0<\delta< ½ (\overset{*}{u}-\xi_1)$. As ξ_2 is a limit point of the sequence $\{u_n\}$, then $\exists$ n_0 such that $\overset{*}{u} <u_{n0}$. Because L is compact and G is cont on L, resulting G is uniformly cont. on L. Thus condition (2) implies that n_0 can be chosen such that,

$u_n - \delta/2 < y_n < u_n + \delta/2$ ……….. (5) and

$Gu_n - \delta/2 < Gy_n < Gu_n + \delta/2$, $\forall$ $n \geq n_0$.

Since $\overset{*}{u} < u_{n0}$, then $\overset{*}{u} < u_n < \overset{*}{u} + \delta$ or $\overset{*}{u} + \delta \leq u_{n0}$.

If $\overset{*}{u} < u_{n0} < \overset{*}{u} + \delta$, then $u_{no} < Gu_{no}$ and resulting $u_{no} \leq y_{no}$.

Which implies $\overset{*}{u} <y_{no} < \overset{*}{u} + \delta$ or $\overset{*}{u} + \delta \leq y_{no}$.

Now assuming $u^* < y_{no} < u^* + \delta$. Then $y_{no} < Gy_{no}$

which gives $u_{no} \le y_{no} < Gy_{no}$. Therefore, $u_{no+1} - u_{no} = \alpha_{no} (Gy_{no} - u_{no}) \ge 0$ or $u^* < u_{no} < u_{no+1}$.

If we take $u^* + \delta \le y_{no}$, we get

$Gu_{no} - \delta/2 \ge y_{no} - \delta/2 \ge u^* + \delta - \delta/2 = u^* + \delta/2$ on account of $y_{no} \le Gu_{no}$.

By (3.2.5), we have $Gy_{no} > Gu_{no} - \delta/2 \ge u^* + \delta/2$

Which additionally to the condition $u^* < u_{no}$ forces us to conclude $u_{no+1} > u^*$.

Eq. (5) gives two cases for the condition $u^* + \delta \le u_{no}$

Case 1 : When $y_{no} \in (u^* + \delta/2, u^* + \delta)$
In this case, $u_{no} < Gy_{no}$ which implies

$$u_{no+1} = (I-\alpha_{no})\, u_{no} + \alpha_{no}\, Gy_{no} - u_{no} \ge \alpha_{no}\, (y_{no} - u_{no})$$

By Eq (5) and $u^* + \delta \le u_{no}$, we get

$$u_{no+1} \ge u_{no} - \alpha_{no}\, \delta/2 \ge u^* + \delta - \delta/2 = u^* + \delta/2 > u^*$$

Case 2 : When $y_{no} \ge u^* + \delta$. Now we faces two possibilities depending upon $Gy_{no} > u_{no}$ or $Gu_{no} < u_{no}$.

If $Gy_{no} > u_{no}$, then $u^* + \delta \le u_{no} + y_{no}$ and application of (5) gives us. $u_{no+1} = u_{no} - \alpha_{no}\, u_{no} + a_{no} + Gy_{no}$

$$\ge u_{no} - \alpha_{no}\, u_{no} + a_{no} + Gu_{no} - \alpha_{no}\, \delta/2$$
$$\ge u_{no} + \alpha_{no}\, (Gu_{no} - u_{no}) - \delta/2 \ge u_{no} - \delta/2$$
$$\ge u^* + \delta - \delta/2 = u^* + \delta/2 > u^*$$

If $Gu_{no} < u_{no}$ then $u^* + \delta \le y_{no} \le u_{no}$. This implies $u_{no+1} \ge u^* + \delta > u^*$ which further gives us $y_{no} \le Gy_{no}$. Also, if $Gy_{no} \le y_{no}$, we have $Gy_{no} < y_{no} < u_{no}$ which ultimately gives $u_{no+1} - u_{no} = \alpha_{no} (Gy_{no} - u_{no}) \le 0$ or $u_{no} \ge u_{no+1}$. Because u^* and δ are positive real numbers. Therefore, we can find a natural number n_1 satisfying $u_{no} \ge u_{no+1} > u^* - n_1\delta$
Now applying this process to $u_{no+1}, u_{no+2}, u_{no+3}$ .... etc. we can prove the existence of a natural number k_0 satisfying the conditions $u^* - k_0\delta > \xi$, and $u_n > u^* - k\delta$, $\forall\ n \ge n_0$. If it is not so, then for any natural number k, we have either $u^* - k\delta \le \xi_1$ or $\exists$ a number $n_k \ge n_0$ such that $u^* - k\delta \ge u_{nk}$. For k=1, $u^* - \delta \le \xi_1$ which is a contradiction of the choice δ to satisfy $2\delta < u^* - \xi_1$ and then the condition $\delta < u^* - \xi_1$.

Thus, the second case bring us with $u^* \ge u_{nk} + k\delta \ge k\delta \ge 0$, $\forall$ k. Because u^* is finite, therefore $\{k\delta\}$ is a bounded sequence, which is a contraction. Therefore there exists at least one k_0 such that $u_n > u^* - k_0\delta > \xi_1$, $\forall\ n \ge n_0$, showing that ξ_1 is not a limit point of $\{u_n\}$ and contradicting $\xi_1 = \lim\inf_n u_n$.

If we take $u_* \in (\xi_1, \xi_2)$ in such a way that $Gu_* < u_*$, we arrive at the conclusion that there exist a $k_1 \in N$ such that $u^* + k_1$ such that ξ_2 and $u_n < u^* - k_1\delta$, $\forall\, n \geq n_0$. This implies ξ_2 is not a limit point of $\{u_n\}$ and contradicts the fact $\xi_2 = \lim \mathrm{Sup}_n\, u_n$. Therefore each point of (ξ_1, ξ_2) is a fixed point of G. This argument along with the continuity of G proves the impossibility of $\xi_1 < G\xi_1$ and $G\xi_2 < \xi_2$ and hence ξ_1 and ξ_2 are not fixed points of G.

Now, by induction method, we shall prove that the sequence $\{u_n\}$ converges to ξ_1 and ξ_2. For this, fix $\varepsilon < ½ (\xi_2 - \xi_1)$. Because G is uniformly cont. and $0 < ½ (\xi_2-\xi_1)$, therefore for any $\in>0$, $\exists$ an $\alpha(\in)>0$ satisfying the condition $| Gx – Gy | < \in$, $\forall\, x, y \in L$ and $|x – y| < \alpha (\in)$........ (6)

Taking $\delta(\in) = \min\{\alpha(\in),\varepsilon\} > 0$. By the second condition of the theorem along with the properties of lim inf, for $\alpha(\in)>0$, $\exists\, n_1 \in N$ such that,

$\xi_1 - \delta(\in) < u_n$ and $\xi_1 - \delta(\in) < y_n$, $n \geq n_1$ .... (7)

and $u_n-\delta(\in) < y_n < u_n + \delta(\in)$

Now define,

$A_\delta = \{n \in N;\ n \geq n_1$ and $u_n, y_n \in (\xi_1-\delta(\in), \xi_1 + \delta(\in)\}$ .... (8)

Because $\xi_1 = \lim\inf u_n$ and from second condition of the theorem, it is very clear that A_δ is non empty. Let n be an arbitary element of A_δ. We need to show that $n+1 \in A_\delta$.

By the definition of A_δ and Eq. (6) along with ξ_1 is a fixed point of G, it follows,

$|Gy_n-u_n| \leq |Gy_n-\xi_1| + |\xi_1-u_n| < \varepsilon + \delta(\varepsilon) < 2\varepsilon$

Hence we have, $|u_{n+1} – u_n| \leq |Gy_n – u_n| \leq 2\varepsilon$. Because $Gu_n \neq u_n$ and $u_n \in (\xi_1-\delta(\in), \xi_1 + \delta(\in))$, therefore, $\xi_1 - \delta(\in) < u_n < \xi_1$ and Eq. (7) gives us $\xi_1-\delta(\in) < u_{n+1}$. Ultimately, by this above argument, $\xi_1 - \delta(\in) < u_{n+1} \leq u_n + 2\varepsilon < \xi_1 + 2\varepsilon$ with $\xi_1 + 2\varepsilon < \xi_2$ on account of $2\varepsilon < \xi_2-\xi_1$. Hence $\xi_1-\delta(\in) < u_{n+1} < \xi_1$+as $\xi_1 \leq u_{n+1} < \xi_1 + 2\varepsilon$ is impossible. Thus, $u_{n+1} \in (\xi_1 - \delta(\in), \xi_1 + \delta(\in))$. Now for y_{n+1}, by Eq. (7), $\xi_1 - \delta (\in) < y_{n+1}$.

Now we are left with, to prove $y_{n+1} < \xi_1 + \delta(\varepsilon)$. By Eq. (7), we get $u_{n+1} - \delta(\in) < y_{n+1} < u_{n+1} + \delta(\in)$ as $n+1>n>n_1$. As $u_{n+1} < \xi_1$, we get $\xi_1-\delta(\in)<y_{n+1} < \xi_1+\delta(\in)$ or $y_{n+1} \in(\xi_1 - \delta(\in), \xi_1 + \delta(\in))$. This implies $n+1 \in A_\delta$ defined by (8) and A_δ is equivalent to N. Hence $|u_n-\xi_1|<\delta(\in) \leq \in$, $\forall n \geq n_1$. Because this inequality is valid for every small $\in > 0$ and $\{u_n\}$ converges to ξ_1.

By the same procedure, $\{u_n\}$ also converges to ξ_2. But the uniqueness of the limit point of the sequence is contracted by $\xi_1 \neq \xi_2$.

$\therefore\ \xi_1 = \xi_2$ and $\{u_n\}$ converges.

Let $a_0 = \xi_1 = \xi_2$, then $Ga_0 = a_0$

Hence the completion of proof

A weak derivation for general Banach spaces given by Rhoades is following.

Theorem 3 : Let K be a non empty closed convex subset of a Banach space. G be a cont. self map of K whose set of fixed points is non empty i.e. $F(G)\neq\phi$.

Let $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences satisfying the following conditions.

1 $0 \leq \alpha_n, \beta_n \leq 1$, $\forall$ n

2 $\lim \beta_n = 0$

3 $\lim\sup \alpha_n > 0$.

If $\{u_{n+1}\}$ converges, then it converges to a fixed point of G, where u_{n+1} is defined as,

$u_{n+1} = (1-\alpha_n)\, u_n + \alpha_n\, G\, [(1–\beta_n)\, u_n + \beta_n\, Gu_n]$, $n \geq 0$

Proof

Let a_0 be a limit point of $\{u_n\}$. Because K is closed and convex, $G(K) \subset K$.

Therefore, $a_0 \in K$. By Eq. (2), we get $\|y_n-u_n\| = \beta_n \| Gu_n-u_n\|$. As G is cont, the sequence $\{Gu_n – u_n\}$ also converges. Second condition of the theorem results, into $\lim \| y_n – u_n\|=\lim \beta_n$, $\lim\|Gu_n – u_n\| = 0$ and therefore $\lim y_n=a_0$, $\lim Gy_n = Ga_0$. Now, we shall prove that $\lim Gy_n=a_0$.

By Equation (2), $\|u_{n+1} – u_n\| = \alpha_n \| Gy_n-u_n\|$.

Now, we get

$$\limsup \|u_{n+1} – u_n\|=\limsup \alpha_n \limsup \| Gy_n-u_n\| = 0$$

Now condition (III) implies that $\lim \|Gy_n-u_n\|=0$

which further implies that a_0 is a fixed point of G.

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