

## Mathematically Examination Of $\Sigma$ -Statistical Convergence

Nadimpally Srinivasulu Goud Research Scholar, Department Of Mathematics, Monad University, Hapur, Uttar Pradesh (India)

Dr. Rajeev Kumar Associate Professor, Department Of Mathematics, Monad University, Hapur, Uttar Pradesh (India)

### Abstract:

In this paper we study one more extension of the concept of statistical convergence namely almost  $\lambda$ -statistical convergence. In section 1.2 we discuss some inclusion relations between almost  $\lambda$ -statistical convergence, strong almost  $(V, \lambda)$ -summability and strong almost convergence. Further in section 1.3 we study the necessary and sufficient condition for an almost statistically convergent sequence to be almost  $\lambda$ -statistically convergent.

[Rani, A. and Kumar, R. STUDIES ON  $\sigma$ -STATISTICAL CONVERGENCE AND LACUNARY  $\sigma$ -STATISTICAL CONVERGENCE. *Rep Opinion* 2021;13(x):-]. ISSN 1553-9873 (print); ISSN 2375-7205 (online). <http://www.sciencepub.net/report>. x. doi:[10.7537/marsroj13xx21xx](https://doi.org/10.7537/marsroj13xx21xx)

**Keywords:** Parameters, Concetroidal matrices, Class 1, Related

### 1.1 Introduction

Let  $s$  be the set of all real or complex sequences and let  $l_\infty$ ,  $c$  and  $c_0$  denote the Banach spaces of bounded, convergent and null sequences  $x = \{\xi_k\}$  respectively normed by  $\|x\| = \sup_k |\xi_k|$ .

Suppose  $D$  is the shift operator on  $s$ , i.e.  $D(\{\xi_k\}) = \{\xi_{k+1}\}$ .

**Definition 1.1.1.** A Banach limit [1] is a linear functional  $L$  defined on  $l_\infty$ , such that

- (i)  $L(x) \geq 0$  if  $\xi_k \geq 0$  for all  $k$ ,
- (ii)  $L(Dx) = L(x)$  for all  $x \in l_\infty$ ,
- (iii)  $L(e) = 1$  where  $e = \{1, 1, 1, \dots\}$ .

**Definition 1.1.2.** A sequence  $x \in l_\infty$  is said to be almost convergent [19] if all Banach limits of  $x$  coincide.

Let  $\hat{c}$  and  $\hat{c}_0$  denote the sets of all sequences which are almost convergent and almost convergent to zero. It was proved by Lorentz [19] that

$$\hat{c} = \{x = \{\xi_k\}: \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \xi_{k+m} \text{ exists uniformly in } m\}.$$

Several authors including Duran [7], King [15] and Lorentz [19] have studied almost convergent sequences.

**Definition 1.1.3.** A sequence  $x = \{\xi_k\}$  is said to be  $(C, 1)$ -summable if and only if  $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \xi_k$  exists.

**Definition 1.1.4.** A sequence  $x = \{\xi_k\}$  is said to be strongly (Cesáro) summable to the number  $\xi$  if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |\xi_k - \xi| = 0.$$

Spaces of strongly Cesáro summable sequences were discussed by Kuttner [17] and some others and this concept was generalized by Maddox [20].

**Remark 1.1.1.** Just as summability gives rise to strong summability, it was quite natural to expect that almost convergence must give rise to a new type of convergence, namely strong almost convergence and this concept was introduced and discussed by Maddox [20].

**Definition 1.1.1.** A sequence  $x = \{\xi_k\}$  is said to be strongly almost convergent to the number  $\xi$  if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |\xi_{k+m} - \xi| = 0 \quad \text{uniformly in } m.$$

If  $[\hat{c}]$  denotes the set of all strongly almost convergent sequences, then

$$[\hat{c}] = \{x = \{\xi_k\}: \text{for some } \xi, \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |\xi_{k+m} - \xi| = 0 \text{ uniformly in } m\}.$$

Let  $\lambda = \{\lambda_n\}$  be a non-decreasing sequence of positive numbers tending to  $\infty$  such that  $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$ .

**Definition 1.1.7.** Let  $x = \{\xi_k\}$  be a sequence. The generalized de la Valée-Pousin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} \xi_k$$

where  $I_n = [n - \lambda_n + 1, n]$ .

**Definition 1.1.8.** A sequence  $x = \{\xi_k\}$  is said to be  $(V, \lambda)$ -summable to a number  $\xi$  [18] if  $t_n(x) \rightarrow \xi$  as  $n \rightarrow \infty$ .

**Remark 1.1.9.** Let  $\lambda_n = n$ . Then  $I_n = [1, n]$  and

$$t_n(x) = \frac{1}{n} \sum_{k=1}^n \xi_k.$$

Hence  $(V, \lambda)$ -summability reduces to  $(C, 1)$ -summability when  $\lambda_n = n$ .

**Definition 1.1.10.** A sequence  $x = \{\xi_k\}$  is said to be strongly almost  $(V, \lambda)$ -summable to a number  $\xi$  if

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} |\xi_{k+m} - \xi| = 0 \quad \text{uniformly in } m.$$

In this case we write  $\xi_k \rightarrow \xi[\hat{V}, \lambda]$  and  $[\hat{V}, \lambda]$  denotes the set of all strongly almost  $(V, \lambda)$ -summable sequences,

i.e.  $[\hat{V}, \lambda] = \{x = \{\xi_k\}: \text{for some } \xi, \lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} |\xi_{k+m} - \xi| = 0 \text{ uniformly in } m\}.$

**Definition 1.1.11.** A sequence  $x = \{\xi_k\}$  is said to be almost statistically convergent to the number  $\xi$  if for each  $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n: |\xi_{k+m} - \xi| \geq \varepsilon\}| = 0 \quad \text{uniformly in } m.$$

In this case we write  $\hat{S}\text{-}\lim \xi_k = \xi$  or  $\xi_k \rightarrow \xi(\hat{S})$  and  $\hat{S}$  denotes the set of all almost statistically convergent sequences.

**Definition 1.1.12.** A sequence  $x = \{\xi_k\}$  is said to be almost  $\lambda$ -statistically convergent to the number  $\xi$  if for each  $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} |\{k \in I_n: |\xi_{k+m} - \xi| \geq \varepsilon\}| = 0 \quad \text{uniformly in } m.$$

In this case we write  $\hat{S}_\lambda\text{-}\lim \xi_k = \xi$  or  $\xi_k \rightarrow \xi(\hat{S}_\lambda)$  and  $\hat{S}_\lambda$  denotes the set of all almost  $\lambda$ -statistically convergent sequences.

**Remark 1.1.13.** If  $\lambda_n = n$ , then  $\hat{S}_\lambda$  is same as  $\hat{S}$ .

## 1.2 SOME INCLUSION RELATION BETWEEN ALMOST $\lambda$ -STATISTICAL CONVERGENCE, STRONG ALMOST $(V, \lambda)$ -SUMMABILITY AND STRONG ALMOST CONVERGENCE

In this section we study some inclusion relations between almost  $\lambda$ -statistical convergence, strong almost  $(V, \lambda)$ -summability and strong almost convergence. First we show that every strongly almost summable sequence is almost statistically convergent.

**Theorem 1.4.1.** If a sequence  $x = \{\xi_k\}$  is almost strongly summable to  $\xi$ , then it is almost statistically convergent to  $\xi$ .

**Proof.** Suppose that  $x = \{\xi_k\}$  is almost strongly summable to  $\xi$ . Then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |\xi_{k+m} - \xi| = 0 \quad \text{uniformly in } m. \quad \dots(1)$$

Let us take some  $\varepsilon > 0$ . We have

$$\begin{aligned} \sum_{k=1}^n |\xi_{k+m} - \xi| &\geq \sum_{\substack{k \in I_n \\ |\xi_{k+m} - \xi| \geq \varepsilon}} |\xi_{k+m} - \xi| \\ &\geq \varepsilon |\{k \leq n : |\xi_{k+m} - \xi| \geq \varepsilon\}| \end{aligned}$$

Consequently,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |\xi_{k+m} - \xi| \geq \varepsilon \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |\xi_{k+m} - \xi| \geq \varepsilon\}|$$

Hence by (1) and the fact that  $\varepsilon$  is fixed number, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |\xi_{k+m} - \xi| \geq \varepsilon\}| = 0 \quad \text{uniformly in } m.$$

$\Rightarrow$   $x$  is almost statistically convergent.

**Theorem 1.2.2.** Let  $\lambda = \{\lambda_n\}$  be same as defined earlier. Then

(i)  $\xi_k \rightarrow \xi[\hat{V}, \lambda] \Rightarrow \xi_k \rightarrow \xi(\hat{S}_\lambda)$

and the inclusion  $[\hat{V}, \lambda] \subseteq \hat{S}_\lambda$  is proper,

(ii) if  $x \in l_\infty$  and  $\xi_k \rightarrow \xi(\hat{S}_\lambda)$ , then  $\xi_k \rightarrow \xi[\hat{V}, \lambda]$  and hence  $\xi_k \rightarrow \xi[\hat{c}]$  provided  $x = \{\xi_k\}$  is not eventually constant.

(iii)  $\hat{S}_\lambda \cap l_\infty = [\hat{V}, \lambda] \cap l_\infty$ ,

where  $l_\infty$  denotes the set of bounded sequences.

**Proof. (i).** Since  $\xi_k \rightarrow \xi[\hat{V}, \lambda]$ , for each  $\varepsilon > 0$ , we have

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} |\xi_{k+m} - \xi| = 0 \quad \text{uniformly in } m. \quad \dots(2)$$

Let us take some  $\varepsilon > 0$ . We have

$$\begin{aligned} \sum_{k \in I_n} |\xi_{k+m} - \xi| &\geq \sum_{\substack{k \in I_n \\ |\xi_{k+m} - \xi| \geq \varepsilon}} |\xi_{k+m} - \xi| \\ &\geq \varepsilon |\{k \in I_n : |\xi_{k+m} - \xi| \geq \varepsilon\}| \end{aligned}$$

Consequently,

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} |\xi_{k+m} - \xi| \geq \varepsilon \lim_{n \rightarrow \infty} \frac{1}{\lambda_n} |\{k \in I_n : |\xi_{k+m} - \xi| \geq \varepsilon\}|$$

Hence by using (2) and the fact that  $\varepsilon$  is fixed number, we have

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} |\{k \in I_n : |\xi_{k+m} - \xi| \geq \varepsilon\}| = 0 \quad \text{uniformly in } m,$$

i.e.  $\xi_k \rightarrow \xi(\hat{S}_\lambda)$ .

It is easy to see that  $[\hat{V}, \lambda] \subsetneq \hat{S}_\lambda$ .

**(ii).** Suppose that  $\xi_k \rightarrow \xi(\hat{S}_\lambda)$  and  $x \in l_\infty$ . Then for each  $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} |\{k \in I_n : |\xi_{k+m} - \xi| \geq \varepsilon\}| = 0 \quad \text{uniformly in } m. \quad \dots(3)$$

Since  $x \in l_\infty$ , there exists a positive real number  $M$  such that  $|\xi_{k+m} - \xi| \leq M$  for all  $k$  and  $m$ . For given  $\varepsilon > 0$ , we have

$$\begin{aligned} \frac{1}{\lambda_n} \sum_{k \in I_n} |\xi_{k+m} - \xi| &= \frac{1}{\lambda_n} \sum_{\substack{k \in I_n \\ |\xi_{k+m} - \xi| \geq \varepsilon}} |\xi_{k+m} - \xi| + \frac{1}{\lambda_n} \sum_{\substack{k \in I_n \\ |\xi_{k+m} - \xi| < \varepsilon}} |\xi_{k+m} - \xi| \\ &\leq \frac{1}{\lambda_n} \sum_{\substack{k \in I_n \\ |\xi_{k+m} - \xi| \geq \varepsilon}} M + \frac{1}{\lambda_n} \sum_{k \in I_n} \varepsilon \end{aligned}$$

$$\begin{aligned}
 &= \frac{M}{\lambda_n} |\{k \in I_n : |\xi_{k+m} - \xi| \geq \varepsilon\}| + \varepsilon \frac{1}{\lambda_n} [n - (n - \lambda_n + 1) + 1] \\
 &= \frac{M}{\lambda_n} |\{k \in I_n : |\xi_{k+m} - \xi| \geq \varepsilon\}| + \varepsilon \frac{1}{\lambda_n} \lambda_n \\
 &= \frac{M}{\lambda_n} |\{k \in I_n : |\xi_{k+m} - \xi| \geq \varepsilon\}| + \varepsilon \\
 \Rightarrow \quad \lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} |\xi_{k+m} - \xi| &\leq M \lim_{n \rightarrow \infty} \frac{1}{\lambda_n} |\{k \in I_n : |\xi_{k+m} - \xi| \geq \varepsilon\}| + \varepsilon
 \end{aligned}$$

Hence by using (3), we get

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} |\xi_{k+m} - \xi| &= 0 \quad \text{uniformly in } m. \quad \dots(4) \\
 \Rightarrow \quad \xi_k &\rightarrow \xi[\hat{V}, \lambda].
 \end{aligned}$$

Further, we have

$$\begin{aligned}
 \frac{1}{n} \sum_{k=1}^n |\xi_{k+m} - \xi| &= \frac{1}{n} \sum_{k=1}^{n-\lambda_n} |\xi_{k+m} - \xi| + \frac{1}{n} \sum_{k=n-\lambda_n+1}^n |\xi_{k+m} - \xi| \\
 &= \frac{1}{n} \sum_{k=1}^{n-\lambda_n} |\xi_{k+m} - \xi| + \frac{1}{n} \sum_{k \in I_n} |\xi_{k+m} - \xi| \\
 &\leq \frac{1}{\lambda_n} \sum_{k=1}^{n-\lambda_n} |\xi_{k+m} - \xi| + \frac{1}{\lambda_n} \sum_{k \in I_n} |\xi_{k+m} - \xi| \\
 &\leq \frac{2}{\lambda_n} \sum_{k \in I_n} |\xi_{k+m} - \xi| \\
 \Rightarrow \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |\xi_{k+m} - \xi| &\leq 2 \lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} |\xi_{k+m} - \xi|
 \end{aligned}$$

Hence

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |\xi_{k+m} - \xi| &= 0 \text{ uniformly in } m. \quad [\text{Using (4)}] \\
 \Rightarrow \quad \xi_k &\rightarrow \xi[\hat{c}].
 \end{aligned}$$

**(iii).** Let  $x \in l_\infty$  be such that  $\xi_k \rightarrow \xi(\hat{S}_\lambda)$ .

Then by (ii),

$$\xi_k \rightarrow \xi[\hat{V}, \lambda].$$

Thus

$$\hat{S}_\lambda \cap l_\infty \subset [\hat{V}, \lambda] \cap l_\infty. \quad \dots(5)$$

Also by (i), we have

$$\xi_k \rightarrow \xi[\hat{V}, \lambda] \Rightarrow \xi_k \rightarrow \xi(\hat{S}_\lambda).$$

So

$$[\hat{V}, \lambda] \subset \hat{S}_\lambda.$$

$$\Rightarrow [\hat{V}, \lambda] \cap l_\infty \subset \hat{S}_\lambda \cap l_\infty. \quad \dots(6)$$

Hence by (5) and (6)

$$\hat{S}_\lambda \cap l_\infty = [\hat{V}, \lambda] \cap l_\infty.$$

This completes the proof of the theorem

### 1.3 NECESSARY AND SUFFICIENT CONDITION FOR AN ALMOST STATISTICALLY CONVERGENT SEQUENCE TO BE ALMOST $\lambda$ -STATISTICALLY CONVERGENT

Since  $\frac{\lambda_n}{n}$  is bounded by 1, we have  $\hat{S}_\lambda \subseteq \hat{S}$  for all  $\lambda$ . In this section we discuss the following relation.

**Theorem 1.4.1.**  $\hat{S} \subseteq \hat{S}_\lambda$  if and only if

$$\liminf_{n \rightarrow \infty} \frac{\lambda_n}{n} > 0, \quad \dots(7)$$

i.e. every almost statistically convergent sequence is almost  $\lambda$ -statistically convergent if and only if (7) holds.

**Proof.** Let us take an almost statistically convergent sequence  $x = \{\xi_k\}$  and assume that (7) holds.

Then for each  $\varepsilon > 0$ , we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n: |\xi_{k+m} - \xi| \geq \varepsilon\}| = 0 \quad \text{uniformly in } m. \quad \dots(8)$$

For given  $\varepsilon > 0$  we get,

$$\{k \leq n: |\xi_{k+m} - \xi| \geq \varepsilon\} \supset \{k \in I_n: |\xi_{k+m} - \xi| \geq \varepsilon\}.$$

Therefore,

$$\begin{aligned} \frac{1}{n} |\{k \leq n: |\xi_{k+m} - \xi| \geq \varepsilon\}| &\geq \frac{1}{n} |\{k \in I_n: |\xi_{k+m} - \xi| \geq \varepsilon\}| \\ &\geq \frac{\lambda_n}{n} \frac{1}{\lambda_n} |\{k \in I_n: |\xi_{k+m} - \xi| \geq \varepsilon\}| \end{aligned}$$

Taking the limit as  $n \rightarrow \infty$  and using (7), we get

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} |\{k \in I_n: |\xi_{k+m} - \xi| \geq \varepsilon\}| = 0 \quad \text{uniformly in } m,$$

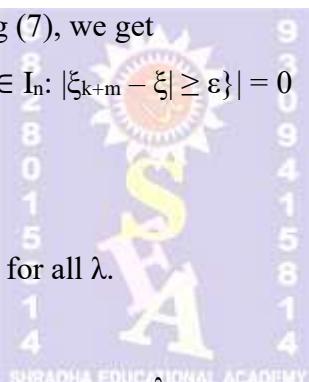
i.e.  $\xi_k \rightarrow \xi(\hat{S}_\lambda)$ .

Hence  $\hat{S} \subseteq \hat{S}_\lambda$  for all  $\lambda$ .

Conversely, suppose that  $\hat{S} \subseteq \hat{S}_\lambda$  for all  $\lambda$ .

We have to prove that (7) holds.

Let us assume that



$$\liminf_{n \rightarrow \infty} \frac{\lambda_n}{n} = 0.$$

As in [9], we can choose a subsequence  $\{n(j)\}$  such that

$$\frac{\lambda_{n(j)}}{n(j)} < \frac{1}{j}.$$

Define a sequence  $x = \{\xi_i\}$  by

$$\xi_i = \begin{cases} 1 & \text{if } i \in I_{n(j)}, j=1,2,3, \dots \\ 0 & \text{otherwise.} \end{cases}$$

Then  $x \in [\hat{c}]$  and hence by Theorem 1.4.1,  $x \in \hat{S}$ . But on the other hand  $x \notin [\hat{V}, \lambda]$  and Theorem 1.4.1 (ii) implies that  $x \notin \hat{S}_\lambda$ . Hence (7) is necessary.

This completes the proof of the theorem

#### References:

- [1] I.J. Maddox, Statistical convergence in a locally convex space, *Math. Proc. Camb. Phil. Soc.* 104 (1988) 141-145.
- [2] Mursaleen, Invariant means and some matrix transformations, *Tamkang J. Math.* 10 (2) (1979) 183-188.
- [3] Mursaleen, Matrix transformations between some new sequence spaces, *Houston J. Math.* 9 (4) (1983) 505-509.
- [4] Mursaleen,  $\lambda$ -statistical convergence, *Math. Slovaca* 50 (1) (2000) 111-115.

- [5] I. Niven and H.S. Zuckerman, *An Introduction to The Theory of Numbers*, Fourth Ed., New York, John Wiley and Sons, 1980.
- [6] T. Šalát, On statistically convergent sequences of real numbers, *Math. Slovaca* 30 (2) (1980) 139-150.
- [7] E. Savas, Some sequence spaces involving invariant means, *Indian J. Math.* 31 (1) (1989) 1-8.
- [8] E. Savas, On lacunary strong  $\sigma$ -convergence, *Indian J. Pure Appl. Math.* 21 (4) (1990) 359-365.
- [9] E. Savas and F. Nuray, On  $\sigma$ -statistical convergence and lacunary  $\sigma$ -statistical convergence, *Math. Slovaca* 43 (1993) 309-315.
- [10] E. Savas, Strong almost convergence and almost  $\lambda$ -statistical convergence, *Hokkaido Math. J.* 29 (2000) 531-536.
- [11] P. Schaefer, Infinite matrices and invariant means, *Proc. Amer. Math. Soc.* 36 (1) (1972) 104-110.
- [12] I.J. Schoenberg, The integrability of certain functions and related summability methods, *Amer. Math. Monthly* 66 (1959) 361-375.

