International Advance Journal of Engineering, Science and Management (IAJESM)
ISSN -2393-8048, July-December 2021, Submitted in July 2021, iajesm2014@gmail.com

A Comparative Study of Fixed-Point Theorems: Exploring Variations in Conditions Within Metric Spaces

Ashesh Kumar Jharwal, Assistant Professor, Shivaji College, University of Delhi, New Delhi

Abstract

An elementary survey of metric spaces, their properties and the concept of continuity as it pertains to them is provided in this work, along with some comparisons to uniform continuity and convergence. An overview of prior research on metric space, its extension, and practical outcomes from studies that utilized these spaces for various purposes are also included. For mixed monotone mappings in partial order metric spaces, we provide a fixed-point theorem, assuming weak contractility. Our theory includes many new findings and is applicable to many different types of problems. In this context, we address the uniqueness of solutions to periodic boundary value problems and talk about whether or not such solutions exist.

Keywords: Comparative Evaluation, Fixed-Point Theorems, Metric Spaces, Auxiliary Theorem, Schauder-Tychonoff Theorem.

1. INTRODUCTION

Mathematicians have given this structure a lot of thought because the fixed-point hypothesis in standard metric space has advanced. The idea of metric space is a powerful tool for addressing specific issues in advanced mathematics. Since the idea of metric spaces has been the subject of much study. After that, only a small number of papers have successfully implemented fixed point hypothesis in those kinds of spaces. Consequently, spaces play a key role in both topology and certain techniques and issues based on logical programming. Metric spaces were initially investigated in 1905 by the eminent French mathematician Maurice Frechet; the term "metric" refers to distance on a number line. In this case, the metric space is defined by Azam et al., which demonstrates that the results of a fixed point on a mapping also follow the rational and defined inequality conditions for that space. After that, theory on cone metric space is generated and initialized. Following that, additional publications have examined the theory's fixed-point generation without verifying it with a complicated highly regarded metric space.

Space scholars have been working diligently lately attempting to furnish the semantics area with an idea of distance. In particular, in his exploration on the denotational semantics of dataflow networks, Matthews laid out the idea of a fractional metric space and found, in addition to other things, a lovely correspondence between these spaces and what are known as weigh table quasimetric spaces. Program check can profit from his exhibit of a speculation of the Banach constriction planning theorem to the fractional metric setting. Fixed point theorems in halfway metric spaces were consequently demonstrated by various scholars.

$$d(Tx, Ty) \leq \emptyset\left(m(x, y)\right)$$

Normal fixed-point consequences of mappings in metric spaces have been acquired utilizing contractive circumstances with the purported correlation capability ϕ of the structure since the renowned work of Boyd and Wong. Capability ϕ was accepted diversely in various articles. These suspicions weren't generally expressed unequivocally, and infrequently they were major areas of strength for excessively. This incorporates a choice of current fixed-point results inside fractional metric spaces.

Reasoning normal fixed-point results in halfway metric spaces and officially expressing various potential circumstances for an examination capability are the objectives of this exploration. These outcomes are similarly relevant to standard metric spaces, since they are excellent circumstances. We will show through models that there are situations where a halfway metric outcome can be utilized rather than the ordinary metric one.

1.1. Objectives of the Study

- To determine which fixed-point theorems for metric spaces are the most important.
- To study the variations of these assumptions that affect the presence of fixed points, for example, contraction constants, completeness, compactness, and others.

2. LITERATURE REVIEW

Ahmed, M. A. (2011) obtained a fixed-point theorem for a summed-up withdrawal in disjoined semi-metric spaces and investigated its implications. In addition to this, we define a single fixed point for each planning and visually represent it. Furthermore, in semi-metric spaces that are totally separated from one another, we present an additional fixed-point theorem. These theorems are a compilation of a number of theorems that may be found in [1-3,5,7,8,11-14,16,19,20,22]. In addition, we would like to make a few remarks concerning [17, Theorem 3] and [21, Theorem 1].

Singh, S. L. (2012) developed a fixed-point theorem for two multivalued maps on a total metric space. This theorem expands upon a previous conclusion of Đorić and Lazović (2011) for a multivalued map on a metric space that satisfies the Æirić-Suzuki-type summed up withdrawal. Moreover, we establish a particular occurrence of a hypothesis of Ìirić's (1974) critical normal fixed-point theorem to be significant. A category of practical conditions that manifest themselves in powerful programming is also examined, along with the question of whether or not these conditions have a characteristic structure.

Senapati, T. (2016) examined an intriguing extension of the classic metric spaces, b-metric spaces, connected metric spaces, and measured spaces as a result of the recent collaborative effort between Jleli plus Samet. Here, we change the outcome for semi-constriction type Čirić mappings and show the same outcome with D-permissible mappings. Two well-known nonlinear withdrawals, namely levelheaded compression mappings and Wardowski type constriction mappings, are also granted fixed point theorems by us. Our findings shed light on various important findings in the text. The right models are provided to support our findings.

Ben-El-Mechaiekh, H. (2013) presented the reader with two of the most foundational set-esteemed maps topological fixed-point theorems, the Browder-Ky Fan and the Kakutani-Ky Fan. It provides a synopsis of the main points, outlining the point's goals, methods, and the most relevant applications. The show is driven by the principles of clarity and directness. Explanation oversimplification is sacrificed for the benefit of applied relevance. We disregard theories based on nuances or wildly varying definitions that can be conveniently limited to conventional contexts unless they are supported by compelling commonsense consequences. Since we think the conventional raised circumstance is where the real lies, we try to steer the conversation in that direction. Due to the brevity of the arguments, this section can serve as an introductory course on topological fixed-point theory and its applications.

Ciepliński (2012) direction of research. A variant of Banach's fixed point theorem was applied by J.A. Bread cook in the year 1991 in order to determine the uniformity of a useful condition in a single variable. It is currently the second most well-known way for demonstrating the Hyers-Ulam strength of helpful circumstances, and this methodology is an example of that. As Radu did, however, the majority of journalists make use of a theorem that was developed by Diaz and Margolis. The basic goal behind this review is to introduce several fixed-point theorems to the hypothesis of the Hyers-Ulam strength of utilitarian conditions. This study will focus on the introduction of these methodologies.

3. PRELIMINARIES

Definition 1: In a fractional metric space, X and p are nonempty sets and p is a halfway metric on X, and that truly intends that for any x, y, and z in X, there exists a capability p : X × X → R.

$$(p_1)\ x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y),$$
$$(p_2)\ p(x,x) \le p(x,y),$$
$$(p_3)\ p(x,y) = p(y,x),$$
$$(p_4)\ p(x,y) \le p(x,z) + p(z,y) - p(z,z).$$

In light of p1 and p2, it is apparent that x = y if px, y = 0. The factors x and px, nonetheless, probably won't approach zero.

The T0 geography τp on X is comprised of the group of open p-balls {Bpx, ε : x ∈ X, ε > 0}, where Bpx, ε = {y ∈ X : px, y < px, x=ε} for all x ∈ X and ε > 0. It is created for each incomplete metric p on X. On the off chance that limn → ∞px, xn px, x, the succession {xn} in X, where p is a component of X, merges to a point x in X concerning τp. This will be composed as xn → x as n approaches limitlessness or as limn → ∞xn = x. For each arrangement {xn} in X, if T: X → X is consistent at x0 ∈ X in τp, then

$$x_n \longrightarrow x_0 \implies Tx_n \longrightarrow Tx_0.$$

Remark: A uniqueness condition is clearly excessive for a grouping's breaking point in a halfway metric space. As a delineation, suppose that X= (0+∞) and p (x, y) and px, y = max {x, y} for all x and y in X. Then, for each {xn} = {1}, p (xn, x=x=p9x, x) and, thusly, xn → 2 and xn → 3 as n → ∞. What's more, the capability p doesn't be guaranteed to must be constant as in pxn, yn → px, y and xn → x and yn → y.
Definition 2: Let X, p be a fractional metric space, as characterized in Definition 2. Then, we have this.
Expecting limn, m → ∞pxn, xm exists and is limited, a grouping {xn} in X, p is alluded to as a Cauchy succession.
2 Expecting that p(x, x) =limn,m → ∞pxn, xm, then, at that point, any Cauchy grouping {xn} in X joins to a point x ∈ X with respect to τp, we say that the space X, p is finished.
Each total fractional metric space has an undeniable full subset. The accompanying capability ps: X × X → R+ is characterized when p is a fractional metric on X.

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

One measure on X is Abstract and Applied Analysis. In addition, the limit limn → ∞psxn, where x = 0 as long as

$$p(x,x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m).$$

4. AUXILIARY THEOREM

Functions Ø:[0,+∞)→[0,+∞] will be examined for the features listed below."ϕn" will represent the nth time iteration of "Ø^n ϕ":

(I) $\varphi(t) < t$ for each $t > 0$ and $\varphi^n(t) \to 0, n \to \infty$ for each $t \geq 0$,
(II) φ is nondecreasing and $\varphi^n(t) \to 0, n \to \infty$ for each $t \geq 0$,
(III) φ is right-continuous, and $\varphi(t) < t$ for each $t > 0$,
(IV) φ is nondecreasing and $\sum_{n\geq 1} \varphi^n(t) < +\infty$ for each $t \geq 0$.

Definition 3: 1. Part II is equivalent to Part I.
2. Nondecreasing ϕ + (III) ⇒ (II).
3. (IV) ⇒ Equation II.
4. Even if ϕ is not decreasing, (III) and (IV) cannot be compared.
Proof:
1. Assume that for all t0 ∈ 0 ∞, there exists an II such that ϕt0 ≥ t0. This means that ϕ2t0 ≥ ϕt0 ≥ t0 if ϕ is monotonic. It is impossible for Ø^n (t_0)→0,n→∞ since continuing by induction we obtain that Ø^2 (t_0)≥ Ø(t_0)≥ t_0 and so on.
2. Suppose that III is true and that ϕ is not decreasing. For every fixed t > 0, the nonincreasing and nonnegative sequence {ϕn(t)} is defined by the monotonicity of ϕ, which means that there exists a limit 〖lim〗 _(n→∞) Ø^n (t)= α≥U. Let us assume that α is greater than zero. Next, we can deduce from III that

$$0 < \alpha \leq \lim_{n\to\infty} \varphi^{n+1}(t) = \lim_{\varphi^n(t)\to\alpha+} \varphi(\varphi^n(t)) = \varphi\left(\lim_{\varphi^n(t)\to\alpha+} \varphi^n(t)\right) = \varphi(\alpha), \qquad (3.1)$$

This contradicts the statement that ϕt is less than t.
3. No brainer.
4. Here is an example to illustrate it.

Illustration: This is where

$$\varphi(t) = \begin{cases} \frac{1}{3}t, & 0 \leq t < 1, \\ \frac{1}{2}, & t = 1, \\ \frac{2}{3}t, & t > 1, \end{cases}$$

meets prerequisites IV however not III. The nondecreasing capability Ø(t)=t/(1+t) meets condition III however not IV since Ø^n (t)=t/(1+nt) doesn't exist. The accompanying lemma and comparative affirmations were used and demonstrated in many articles to make different fixed statement results.

Explanation: Characterize X as a metric space of aspect d and consider {yn} as a succession in X fulfilling the condition that {d(y_(n+1),y_n)} is nonincreasing.

$$\lim_{n\to\infty} d(y_{n+1}, y_n) = 0.$$

If the series {y2n} is not Cauchy, then there is a positive integer ε and two positive integer subsequences {mk} and {nk} with the end goal that the accompanying four arrangements approach ε = 0 as k methodologies vastness:

$$d(y_{2m_k}, y_{2n_k}), \qquad d(y_{2m_k}, y_{2n_k+1}), \qquad d(y_{2m_k-1}, y_{2n_k}), \qquad d(y_{2m_k-1}, y_{2n_k+1}).$$

5. THE SCHAUDER-TYCHONOFF FIXED POINT THEOREM

Definition 4: In a Banach space X and a Banach space Y, where C is a subspace of X, a guide f: C → Y is viewed as smaller if and provided that it decreases limited sets to reduced ones somewhat. On the off chance that f is an individual from L(X, Y), it is comparable to saying that the picture of the unit shut ball under f is moderately smaller. As far as groupings, f is conservative if and provided that the succession f(xn) has a merged aftereffect for each limited grouping xn.

Theorem [Schaefer]: Let X be a Banach space and f: X → X be nonstop and minimized. Expect further that the set has limits. When this is valid, f has a fixed point.

$$F = \{x \in X \; : \; x = \lambda f(x) \quad for\ some \quad \lambda \in [0,1]\}$$

Remark: If we are able to demonstrate a priori estimations on the set of all potential fixed points of λf, then the theorem above will be true. Partial differential equations are a common use of this method, which entails estimating a solution to an equation and then using those estimates to establish that the solution exists.

proof Let the map be defined as r> 〖sup〗_(x∈F) ||x||.

$$g(x) = \begin{cases} f(x) & \text{if} \quad \|f(x)\| \leq 2r \\ \dfrac{2r f(x)}{\|f(x)\|} & \text{if} \quad \|f(x)\| > 2r. \end{cases}$$

Consequently, the function g: BX(0, 2r) → BX(0, 2r) is compact and continuous. The existence of an element x0 ∈ BX(0, 2r) such that g(x0) = x0 is proven by Theorem 1.18. If kf(x0)k is less than or equal to 2r, then

$$x_0 = \lambda_0 f(x_0) \qquad \text{with} \qquad \lambda_0 = \frac{2r}{\|f(x_0)\|} < 1$$

Despite x0 being a member of F, this fact is used to force ||x0|| = 2r. Thus, we obtain that g(x0) = f(x0) = x0.

6. CONCLUSION

In this study, fixed-point theorems in metric spaces are compared and evaluated. Using uniform continuity as a comparison, it explains the fundamentals of metric spaces, convergence, and continuity. Previous work on metric spaces, as well as generalizations of these spaces and practical applications of these conclusions, are also reviewed in the study. On a metric space with halfway request, the creators show a fixed-point theorem for a blended droning planning under the frail contractility condition. Among many other things, the theorem can be used to find and prove that periodic boundary value problems have unique solutions. Finding the best fixed-point theorems for metric spaces and looking at how different circumstances affect the presence of fixed points are the main goals of this research.

REFERENCES

1. *Abdeljawad, T. (2013). Meir-Keeler a-contractive fixed and common fixed-point theorems. Fixed point theory and applications, 2013, 1-10.*
2. *Ahmed, M. A. (2011). Some fixed point theorems. International Journal of Geometric Methods in Modern Physics, 8(01), 1-8.*
3. *Al-Mezel, S. A. R., Al-Solamy, F. R. M., & Ansari, Q. H. (Eds.). (2014). Fixed point theory, variational analysis, and optimization. CRC press.*

4. *Ansari, A. H., Chandok, S., & Ionescu, C. (2014). Fixed point theorems on b-metric spaces for weak contractions with auxiliary functions. Journal of Inequalities and Applications, 2014, 1-17.*
5. *Ben-El-Mechaiekh, H. (2013). Some fundamental topological fixed point theorems for set-valued maps. In Topics in Fixed Point Theory (pp. 237-272). Cham: Springer International Publishing.*
6. *Bigi, G., & Passacantando, M. (2014). A close look at auxiliary problem principles for equilibria.*
7. *Ciepliński, K. (2012). Applications of fixed-point theorems to the Hyers-Ulam stability of functional equations–a survey. Annals of Functional Analysis, 3(1), 151-164.*
8. *Farmakis, I., & Moskowitz, M. (2013). Fixed point theorems and their applications. World Scientific.*
9. *Gil, M., & Wędrychowicz, S. (2013). Schauder-Tychonoff Fixed-Point Theorem in Theory of Superconductivity. Journal of Function Spaces, 2013(1), 692879.*
10. *Hu, X. Q., Zheng, M. X., Damjanović, B., & Shao, X. F. (2013). Common coupled fixed point theorems for weakly compatible mappings in fuzzy metric spaces. Fixed Point Theory and Applications, 2013, 1-11.*
11. *Liu, Z., Li, X., Kang, S. M., & Cho, S. Y. (2011). Fixed point theorems for mappings satisfying contractive conditions of integral type and applications. Fixed point theory and Applications, 2011, 1-18.*
12. *Park, S. E. H. I. E. (2015). Recent applications of some analytical fixed point theorems. Nonlinear Analysis and Convex Analysis (NACA 2015, Chiang Rai), 259-273.*
13. *Senapati, T., Dey, L. K., & Dolićanin-Đekić, D. (2016). Extensions of Ćirić and Wardowski type fixed point theorems in D-generalized metric spaces. Fixed Point Theory and Applications, 2016, 1-14.*
14. *Singh, S. L., Mishra, S. N., Chugh, R., & Kamal, R. (2012). General common fixed-point theorems and applications. Journal of applied mathematics, 2012.*
15. *Van An, T., Van Dung, N., Kadelburg, Z., & Radenović, S. (2015). Various generalizations of metric spaces and fixed-point theorems. Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas, 109, 175-198.*